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Inverses of Perron complements of inverse M-matrices

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Abstract

The concept of the *Perron complement* of a nonnegative and irreducible matrix was introduced by Meyer in 1989 and it was used by him to construct an algorithm for computing the stationary distribution vector for Markov chains. Here we consider properties of the Perron complement of an $n \times n$ matrix K which is an inverse of an irreducible M-matrix. We first show that the Perron complements of K are inverses of M-matrices and that the inverses of associated principal submatrices of K are sandwiched between the inverses of the Perron complements of K and the inverses of the corresponding Schur complements of K. We then investigate the directed graph of the inverse of the Perron complements of such matrices K.

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1. Introduction

Let $K \in \mathbb{R}^{n,n}$ be the space of all real $n \times n$ matrices and let γ and δ be nonempty ordered subsets of $\langle n\rangle := \{1, \ldots, n\}$, both of strictly increasing integers. By $K[\gamma, \delta]$ we shall denote the submatrix of K whose rows and columns are determined by γ and δ, respectively. In the special case when $\gamma = \delta$, we shall use $K[\gamma]$ to denote $K[\gamma, \gamma]$, the principal submatrix of K based on γ.

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Suppose that $\beta \subset \langle n\rangle$. If $K[\beta]$ is nonsingular, then the Schur complement of $K[\beta]$ in K is given by

$$\mathscr{S}(K/K[\beta]) \;=\; K[\alpha] - K[\alpha,\beta](K[\beta])^{-1}K[\beta,\alpha], \tag{1.1}$$

where $\alpha = \langle n\rangle\backslash\beta$. If $K \in \mathbb{R}^{n,n}$ is a nonsingular M-matrix, then all its principal submatrices are invertible. A well known result due to Crabtree [3] states that if K is a nonsingular M-matrix, then (all) its Schur complements are M-matrices. If, however, K is an inverse of an M-matrix, then again, as is well known all its principal submatrices are invertible and the inverses of its Schur complements are M-matrices by virtue of their being principal submatrices of K^{-1}. The inverses of the principal submatrices of K are M-matrices because they are Schur complements of K^{-1} (see display in (2.4)).

For more background material on nonnegative matrices, M-matrices, and directed graphs we refer the reader to the book by Berman and Plemmons [1]. For background material on matrix theory, linear algebra, and matrix computations see the books by Horn and Johnson [4] and Golub and van Loan [5].

In connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer [9,10] introduced, for an $n \times n$ nonnegative and irreducible matrix K, the notion of the Perron complement. Again, let $\beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle\backslash\beta$. Then the Perron complement of $K[\beta]$ in K is given by

$$\mathscr{P}(K/K[\beta]) \;=\; K[\alpha] + K[\alpha,\beta](\rho(K)I - K[\beta])^{-1}K[\beta,\alpha], \tag{1.2}$$

where $\rho(\cdot)$ denotes the *spectral radius* of a matrix. Recall that as K is irreducible, $\rho(K) > \rho(K[\beta])$, so that the expression on the right-hand side of (1.2) is well defined. Meyer has derived several interesting and useful properties of $\mathscr{P}(K/K[\beta])$. The first is that $\rho\left(\mathscr{P}(K/K[\beta])\right) = \rho(K)$. The second is that if K is row stochastic, then so is $\mathscr{P}(K/K[\beta])$. In the latter case, Meyer has shown how, if $\beta_1,\ldots,\beta_s$ are disjoint subsets whose union is $\langle n\rangle$, then the stationary distribution vector for the (entire) Markov process can be aggregated from the stationary distribution vectors of its Perron complements $\mathscr{P}(K/K[\beta_1]),\ldots,\mathscr{P}(K/K[\beta_s])$.

It is seen that in the case when K is reducible, not necessarily for every $\beta \subset \langle n\rangle$, the Perron complement exists since $\rho(K)$ may equal $\rho(K[\beta])$. Thus to avoid any difficulties, we shall always assume that K is irreducible. Since our starting point here will be that K is an inverse of an irreducible M-matrix, K will always be, in fact, positive. We comment that for general irreducible nonnegative $n \times n$ matrices K, Johnson and Xenophotos [6] investigate when the Perron complements are primitive or just irreducible and thus answer some issues which were raised by Meyer in his earlier paper.

In this paper we shall first show, in Section 2, that if K is an inverse of an irreducible M-matrix, then its Perron complements are (also) inverses of M-matrices. In fact, we shall work with a slight extension of the notion of the Perron complement. For any $\beta \subset \langle n\rangle$ and for any $t \geqslant \rho(K)$, let the *extended Perron complement at t* be the matrix

$$\mathscr{P}_t(K/K[\beta]) := K[\alpha] + K[\alpha,\beta](tI - K[\beta])^{-1}K[\beta,\alpha], \tag{1.3}$$

which continues to be well defined since $t \geqslant \rho(K) > \rho(K[\beta])$. We shall obtain that $\mathscr{P}_t(K/K[\beta])$ is an inverse of an M-matrix and that

$$(\mathscr{P}_t(K/K[\beta]))^{-1} \leqslant (K[\alpha])^{-1} \leqslant (\mathscr{S}(K/K[\beta]))^{-1}, \tag{1.4}$$

where, as before, $\alpha = \langle n\rangle\backslash\beta$, with $(\mathscr{P}_t(K/K[\beta]))^{-1}$ being an entrywise increasing matrix in $[\rho(K), \infty)$ for which

$$\lim_{t\to\infty} (\mathscr{P}_t(K/K[\beta]))^{-1} = (K[\alpha])^{-1}.$$

We can thus view the M-matrix $(K[\alpha])^{-1}$ as separating between the M-matrices $(\mathscr{P}_t(K/K[\beta]))^{-1}$ and $(\mathscr{S}(K/K[\beta]))^{-1}$.

Let $\Gamma(\cdot)$ denote the *directed graph of a matrix* (for the definition see, for example, [1, Definition 2.4]). Continuing with the assumption that $K \in \mathbb{R}^{n,n}$ is an inverse of an irreducible M-matrix, $\beta \subset \langle n\rangle$, $\alpha = \langle n\rangle\backslash\beta$, and $t \in [\rho(K), \infty)$, we shall show in Section 3 that

$$\Gamma\left((\mathscr{S}(K/K[\beta]))^{-1}\right) \subseteq \Gamma\left((K[\alpha])^{-1}\right) = \Gamma\left((\mathscr{P}_t(K/K[\beta]))^{-1}\right). \tag{1.5}$$

Several corollaries follow and we provide a few examples to illustrate our results.

Throughout the paper, for brevity in our proofs, we shall adopt the following notations: if $K \in \mathbb{R}^{n,n}$, $\beta \in \langle n\rangle$, and $\alpha = \langle n\rangle\backslash\beta$, then

$$\begin{aligned} A &= K[\alpha],\\ B &= K[\alpha,\beta],\\ C &= K[\beta,\alpha],\\ D &= K[\beta,\beta]. \end{aligned} \tag{1.6}$$

We further easily observe from (1.2) that on letting $\tilde{K} = K/\rho(K)$, $\rho(\tilde{K}) = 1$ and

$$\mathscr{P}_{t/\rho(K)}(\tilde{K}/\tilde{K}[\beta]) = \frac{\mathscr{P}(K/K[\beta])}{\rho(K)}$$

for all $\beta \subset \langle n\rangle$. This means that in our proofs we can assume, without loss of generality, that $\rho(K) = 1$.

2. Perron complements of inverse M-matrices are inverse M-matrices

We can begin immediately with our first main result:

Theorem 2.1. *Let $K \in \mathbb{R}^{n,n}$ be an inverse of an irreducible M-matrix and let $\beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle\backslash\beta$. Then for any $t \in [\rho(K), \infty)$, the matrix*

$$\mathscr{P}_t\,(K/K[\beta]) = K[\alpha] - K[\alpha,\beta](tI - K[\beta])^{-1}K[\beta,\alpha] \tag{2.1}$$

is invertible and its inverse is an M-matrix. In particular, the Perron complement $\mathscr{P}(K/K[\beta])$ $(= \mathscr{P}_1(K/K[\beta]))$ is an inverse of an M-matrix.

Proof. For ease of notation we adopt the substitutions in (1.6). We also recall that, without loss of generality, we can assume that $\rho(K) = 1$. We may further assume with no restrictions on the conclusion that K has been symmetrically permuted to the block form:

$$K = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We begin by showing that, for $t \geqslant 1$, the matrix

$$\mathscr{P}_t(K/D) = A + B(tI - D)^{-1}C \tag{2.2}$$

is invertible and by computing its inverse. This we do using a consequence of the Woodbury formula (see, e.g., [2, (3.1.6), p. 129]) from which we see that, if $E \in \mathbb{C}^{m,m}$ and $F \in \mathbb{C}^{k,k}$ are invertible matrices and if $U \in \mathbb{C}^{m,k}$ and $V \in \mathbb{C}^{k,m}$ are matrices for which the matrix $F^{-1} + VE^{-1}U$ is invertible, then the matrix $E + UFV$ is invertible and its inverse is given by:

$$(E + UFV)^{-1} = E^{-1} - E^{-1}U(F^{-1} + VE^{-1}U)^{-1}VE^{-1}. \tag{2.3}$$

Now since K is invertible, its inverse in block form is given by

$$K^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(K/A)^{-1}CA^{-1} & -A^{-1}B(K/A)^{-1} \\ -(K/A)^{-1}CA^{-1} & (K/A)^{-1} \end{pmatrix}. \tag{2.4}$$

Moreover, since K^{-1} is an M-matrix we must have that the

$$-(K^{-1})_{1,2} = A^{-1}B(K/A)^{-1} = A^{-1}B(D - CA^{-1}B)^{-1} \geqslant 0.$$

But $(K/A)^{-1}$ is a principal submatrix of an M-matrix and hence an M-matrix itself which readily implies, because of its monotonicity, that $A^{-1}B$ is a nonnegative matrix. In a similar way we show that CA^{-1} is also a nonnegative matrix, but in any case, we now have that $CA^{-1}B \geqslant 0$. Now, the matrix $\mathscr{S}(K/A) = D - CA^{-1}B$ is an inverse of a principal submatrix of the M-matrix K^{-1} and hence it is nonnegative so that in fact we have that $D \geqslant D - CA^{-1}B \geqslant 0$. But then, from the Perron–Frobenius theory we deduce that

$$1 = \rho(K) > \rho(D) \geqslant \rho(D - CA^{-1}B). \tag{2.5}$$

This implies that the matrix $F := tI - (D - CA^{-1}B)$ is invertible whenever $t \geqslant 1$. Certainly the matrix $E := A$ is invertible since A is an inverse of a Schur complement of K^{-1} and hence an inverse of a nonsingular M-matrix. Letting $U = B$ and $V = C$ we obtain that not only is $\mathscr{P}(K/D)$ invertible, but a substitution in (2.3) yields that

$$(\mathscr{P}_t([K/D]))^{-1} = A^{-1} - A^{-1}B\left[tI - \left(D - CA^{-1}B\right)\right]^{-1}CA^{-1}. \tag{2.6}$$

Finally, because of (2.5), $[tI - (D - CA^{-1}B)]^{-1}$ is an inverse M-matrix and so a nonnegative matrix. Thus as A^{-1} is an M-matrix, $A^{-1}B \geqslant 0$, and $CA^{-1} \geqslant 0$, we see that $(\mathscr{P}_t([K/D]))^{-1}$ is an invertible matrix whose off-diagonal entries are nonpositive and whose inverse is nonnegative and hence it is an M-matrix. This completes our proof. □

We now have the following corollary in which we show that under the assumptions in Theorem 2.1, the inverse of $K[\alpha]$ is sandwiched between the inverse of the extended Perron complement $\mathscr{P}_t(K/K[\beta])$ and the inverse of the corresponding Schur complement $\mathscr{S}(K/K[\beta])$.

Corollary 2.2. *Let $K \in \mathbb{R}^{n,n}$ be an inverse of an irreducible M-matrix and let $\beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \backslash \beta$. Then for any $t \in [\rho(K), \infty)$, the following ordering holds between the three M-matrices $(\mathscr{P}_t(K/K[\beta]))^{-1}$, $(K[\alpha])^{-1}$, and $(\mathscr{S}(K/K[\beta]))^{-1}$:*

$$(\mathscr{P}_t(K/K[\beta]))^{-1} \leqslant (K[\alpha])^{-1} \leqslant (\mathscr{S}(K/K[\beta]))^{-1}. \tag{2.7}$$

Moreover, as function of t, the matrix $(\mathscr{P}_t(K/K[\beta]))^{-1}$ is entrywise increasing in $[\rho(K), \infty)$ and

$$\lim_{t\to\infty} (\mathscr{P}_t(K/K[\beta]))^{-1} = (K[\alpha])^{-1}. \tag{2.8}$$

Proof. Again, for ease of notation we adopt the substitutions in (1.6). We also recall that, without loss of generality, we can assume that $\rho(K) = 1$.

From Section 1 and Theorem 2.1 we already know that all three matrices appearing in (2.7) are M-matrices. The left inequality in 2.7 is immediate from (2.6) and the fact established in the previous proof that $A^{-1}B\left[tI - (D - CA^{-1}B)\right]^{-1} CA^{-1} \geqslant 0$. Consider now the invertible matrix $\mathscr{S}(K/D)$, explicitly given in (1.1). Applying the consequence of Woodbury's formula given in (2.3) to obtain $(\mathscr{S}(K/D))^{-1}$ yields that

$$(\mathscr{S}(K/D))^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}. \tag{2.9}$$

We know from the proof of Theorem 2.1 that the matrix

$$N := A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} \geqslant 0. \tag{2.10}$$

However, both A^{-1} and $(\mathscr{S}(K/D))^{-1}$ are M-matrices which yield the right inequality in (2.7).

Suppose now the $t_2 \geqslant t_1 \geqslant \rho(K)$ so that, as M-matrices,

$$t_2 I - (D - CA^{-1}B) \geqslant t_1 I - (D - CA^{-1}B).$$

Then by the theory of M-matrices,

$$\left(t_1 I - \left(D - CA^{-1}B\right)\right)^{-1} \geqslant \left(t_2 I - \left(D - CA^{-1}B\right)\right)^{-1}$$

and so $\left(\mathscr{P}_{t_2}(K/K[\beta])\right)^{-1} \geqslant \left(\mathscr{P}_{t_1}(K/K[\beta])\right)^{-1}$ by (2.6) and because $A^{-1}B \geqslant 0$ and $CA^{-1} \geqslant 0$ as established earlier. This shows that as a function of t, the matrix $(\mathscr{P}_t(K/K[\beta]))^{-1}$ is entrywise increasing in $[\rho(K), \infty)$. Finally, as

$$\lim_{t\to\infty} \left[tI - \left(D - CA^{-1}B\right)\right]^{-1} = 0,$$

(2.8) readily follows from (2.6). □

3. The directed graph of $(\mathscr{P}_t(K/K[\cdot]))^{-1}$

In this section, we investigate the directed graph of the inverses of the extended Perron complements and compare them, for example, to the directed graphs of the inverses of the corresponding Schur complements. Our main result is as follows.

Theorem 3.1. *Let $K \in \mathbb{R}^{n,n}$ be an inverse of an irreducible M-matrix and let $\beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle\backslash\beta$. Then for any $t \in [\rho(K), \infty)$,*

$$\Gamma((\mathscr{S}(K/K[\beta]))^{-1}) \subseteq \Gamma((K[\alpha])^{-1}) = \Gamma((\mathscr{P}_t(K/K[\beta]))^{-1}). \tag{3.1}$$

Proof. For ease of notation, we shall again adopt the substitutions in (1.6). We also recall that, without loss of generality, we can assume that $\rho(K) = 1$.

We have shown in the proof of Corollary 2.2 and in the proof of Theorem 2.1 leading to the corollary that both $(\mathscr{S}(K/D))^{-1}$ and A^{-1} are M-matrices and $A^{-1} \leqslant (\mathscr{S}(K/D))^{-1}$. This means that for $i \neq j$,

$$((\mathscr{S}(K/D))^{-1})_{i,j} \neq 0 \implies (A^{-1})_{i,j} \neq 0.$$

Thus, clearly, $\Gamma((\mathscr{S}(K/D))^{-1}) \subseteq \Gamma(A^{-1})$ since the diagonal entries of both $(\mathscr{S}(K/D))^{-1}$ and A^{-1} are nonzero. This establishes one part of (3.1).

Next, note that (2.10) in conjunction with $A^{-1} \leqslant (\mathscr{S}(K/D))^{-1}$, says, that in particular, for $i \neq j$,

$$(A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1})_{i,j} \neq 0 \implies (A^{-1})_{i,j} \neq 0$$

so that, as all the diagonal entries of A^{-1} are nonzero, we must also have that

$$\Gamma(A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}) \subseteq \Gamma(A^{-1}). \tag{3.2}$$

Recall now that $(D - CA^{-1}B)^{-1} = (\mathscr{S}(K/A))^{-1}$ is an M-matrix showing, by Lewin and Neumann [8, Corollary 1, p. 45], that the directed graph of the nonnegative matrix $\mathscr{S}(K/A)$ must satisfy that

$$\Gamma(\mathscr{S}(K/A)) = \overline{\Gamma(\mathscr{S}(K/A))},$$

where, to recall, $\Gamma(\cdot)$ denotes the directed graph of a matrix and where $\overline{\Gamma(\cdot)}$ denotes the *transitive closure of a graph* (see [11, Section 2]). Observe that the matrix N explicitly given on the right-hand side of (2.10) admits the following factorization into a product of three nonnegative matrices:

$$N = \underbrace{A^{-1}B\,(\mathscr{S}(K/A))^{-1}}_{\geqslant 0}\,\underbrace{\mathscr{S}(K/A)}_{\geqslant 0}\,\underbrace{(\mathscr{S}(K/A))^{-1}\,CA^{-1}}_{\geqslant 0}. \tag{3.3}$$

Consider now the matrix

$$J := A^{-1}B[tI - (D - CA^{-1}B)]^{-1}CA^{-1},$$

which forms, up to a sign, the second term on the right-hand side of (2.6). We see that J admits the factorization

$$J = \underbrace{A^{-1} B\, (\mathscr{S}(K/A))^{-1}}_{\geqslant 0}$$
$$\times \underbrace{\left[t\, (\mathscr{S}(K/A))^{-2} - (\mathscr{S}(K/A))^{-1}\right]^{-1}}_{\text{directed graph yet to be determined}} \underbrace{(\mathscr{S}(K/A))^{-1}\, C A^{-1}}_{\geqslant 0}. \tag{3.4}$$

Now, according to Schneider [11, Lemma 2.2], for any $n \times n$ nonsingular matrix L,

$$\Gamma\left(L^{-1}\right) \subseteq \overline{\Gamma(L)}.$$

Thus for the middle factor in (3.4) we have that

$$\Gamma\left(\left[t\, (\mathscr{S}(K/A))^{-2} - (\mathscr{S}(K/A))^{-1}\right]^{-1}\right)$$
$$\subseteq \overline{\Gamma\left(t\, (\mathscr{S}(K/A))^{-2} - (\mathscr{S}(K/A))^{-1}\right)} \subseteq \Gamma\left(\mathscr{S}(K/A)\right). \tag{3.5}$$

The last containment follows because both $t\,(\mathscr{S}(K/A))^{-2}$ and $(\mathscr{S}(K/A))^{-1}$ are polynomials in $\mathscr{S}(K/A)$ which imply, as $\mathscr{S}(K/A)$ is a nonnegative matrix whose directed graph coincides with its closure, that $\Gamma\left((\mathscr{S}(K/A))^{\ell}\right) = \Gamma\left(\mathscr{S}(K/A)\right)$, for all $\ell \geqslant 1$. Comparing (3.3), (3.4), and (3.5) we see that

$$\Gamma(J) \subseteq \Gamma(N). \tag{3.6}$$

But from (3.2), we know that $\Gamma(N) \subseteq \Gamma(A^{-1})$. However, we have already observed earlier that $\Gamma(A^{-1}) \subseteq \Gamma(A^{-1} - N)$. From all these it follows that $\Gamma((\mathscr{P}_t(K/D))^{-1}) \subseteq \Gamma(A^{-1})$ and our proof is done. $\square$

We now give a few examples illustrating the results in Theorem 3.1. We begin by recalling the following theorem due to Lewin which characterizes the class of the inverse tridiagonal M-matrices.

Theorem 3.2 [7, Theorem 1]. *Let $K \in \mathbb{R}^{n,n}$. Consider the following three conditions*:
(i) *K is nonsingular and totally nonnegative.*
(ii) *K is nonsingular and K^{-1} is an M-matrix.*
(iii) *K is nonsingular and K^{-1} is tridiagonal.*
Then any two of the three conditions imply the third.

For the class of inverse tridiagonal M-matrices, we thus have the following result.

Corollary 3.3. *Let $K \in \mathbb{R}^{n,n}$ be an inverse of an irreducible tridiagonal M-matrix. Then for any subset $\beta \subseteq \langle n \rangle$, the Perron complement*

$$\mathscr{P}(K/K[\beta]) = K[\alpha] + K[\alpha, \beta]\, (I - K[\beta])^{-1}\, K[\beta, \alpha], \tag{3.7}$$

where $\alpha = \langle n \rangle \backslash \beta$, *is an inverse of an irreducible tridiagonal M-matrix and hence* (*also*) *totally nonnegative.*

Proof. As before, let $D = K[\beta]$ and $A = K[\alpha]$. Then the matrix A, being a principal submatrix of a totally nonnegative matrix is, itself, totally nonnegative. But then as A is also an inverse of an M-matrix now implies, by Theorem 3.2 that A^{-1} is a tridiagonal M-matrix. But then by Theorem 3.1, $(\mathscr{P}(K/D))^{-1}$ is a tridiagonal M-matrix. The final part of the corollary follows by applying Theorem 3.2 to the tridiagonal M-matrix $(\mathscr{P}(K/D))^{-1}$. $\square$

As a concrete example consider the inverse tridiagonal M-matrix:

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}^{-1}$$

and let $\beta = \{2, 3, 6\}$ and $\alpha = \{1, 4, 5\}$. Then

$$\begin{aligned} \mathscr{P}(K/K[\beta]) &= K[\alpha] + K[\alpha, \beta](I - K[\beta])^{-1} K[\beta, \alpha] \\ &= \begin{bmatrix} 1.3491 & 2.0858 & 2.2229 \\ 2.0858 & 7.5086 & 8.0024 \\ 2.2229 & 8.0024 & 9.5944 \end{bmatrix} \end{aligned}$$

and we find that

$$(\mathscr{P}(K/K[\beta]))^{-1} = \begin{bmatrix} 1.2992 & -0.36089 & 0 \\ -0.36089 & 1.2992 & -1 \\ 0 & -1 & 0.93830 \end{bmatrix}$$

and

$$(K[\alpha])^{-1} = \begin{bmatrix} 4/3 & -1/3 & 0 \\ -1/3 & 4/3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and we see that $\Gamma\left((\mathscr{P}(K/K[\beta]))^{-1}\right) = \Gamma\left((K[\alpha])^{-1}\right)$. On the other hand, we know that

$$(\mathscr{S}(K/K[\beta]))^{-1} = (K^{-1})[\alpha] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

from which we see that *strict containment* relation can hold on the left-hand side of (3.1). If, however, we choose $\beta = \{4, 5, 6\}$, then $(\mathscr{S}(K/K[\beta]))^{-1} = K^{-1}[\{1, 2, 3\}]$ which is an irreducible tridiagonal matrix. Since both $(K[\{1, 2, 3\}])^{-1}$ and $(\mathscr{P}_t(K/K[\{1, 2, 3\}]))^{-1}$ must, by Corollary 3.3, be inverses of irreducible tridiagonal M-matrices, we see that in this case *equality* holds throughout (3.1).

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References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadelphia, PA, 1994.
[2] A. Björck, Numerical Methods for Least Squares Problems, SIAM, Philadelphia, PA, 1966.
[3] D.E. Crabtree, Applications of M-matrices to nonnegative matrices, Duke Math. J. 33 (1966) 197–208.
[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.
[5] G. Golub, C. Van Loan, Matrix Computation, third ed., The John Hopkins University Press, Baltimore, MD, 1996.
[6] C.R. Johnson, C. Xenophotos, Irreducibility and primitivity of Perron complements: applications of the compressed graph, in: Graph Theory and Sparse Matrix Computations, IMA Vol. Math. Appl., vol. 56, 1993, pp. 101–106.
[7] M. Lewin, Totally nonnegative, M-, and Jacobi matrices, SIAM J. Algebraic Discrete Methods 4 (1980) 419–421.
[8] M. Lewin, M. Neumann, On the inverse M-matrix problem for $(0, 1)$-matrices, Linear Algebra Appl. 30 (1980) 41–50.
[9] C.D. Meyer, Uncoupling the Perron eigenvector problem, Linear Algebra Appl. 114/115 (1989) 69–94.
[10] C.D. Meyer, Stochastic complementation, uncoupling Markov chains, and the theory of nearly reducible systems, SIAM Rev. 31 (1989) 240–272.
[11] H. Schneider, Theorems on M-splittings of a singular M-matrix which depend on graph structure, Linear Algebra Appl. 58 (1984) 407–424.